

## Computing the vertex Folkman numbers

$$F_v(a_1, \dots, a_s; m - 1)$$

when  $\max\{a_1, \dots, a_s\} = 6$

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**Abstract**

Let  $G$  be a graph and  $a_1, \dots, a_s$  be positive integers. Then  $G \xrightarrow{v} (a_1, \dots, a_s)$  means that for every coloring of the vertices of  $G$  in  $s$  colors there exists  $i \in \{1, \dots, s\}$ , such that there is a monochromatic  $a_i$ -clique of color  $i$ . The vertex Folkman number  $F_v(a_1, \dots, a_s; q)$  is defined by the equality:

$$F_v(a_1, \dots, a_s; q) = \min\{|V(G)| : G \xrightarrow{v} (a_1, \dots, a_s) \text{ and } K_q \not\subseteq G\}.$$

Let  $m = \sum_{i=1}^s (a_i - 1) + 1$ . It is easy to see that  $F_v(a_1, \dots, a_s; q) = m$  if  $q \geq m + 1$ . In [11] it is proved that  $F_v(a_1, \dots, a_s; m) = m + \max\{a_1, \dots, a_s\}$ . We know all the numbers  $F_v(a_1, \dots, a_s; m - 1)$  when  $\max\{a_1, \dots, a_s\} \leq 5$  and none of these numbers is known if  $\max\{a_1, \dots, a_s\} \geq 6$ . In this paper we compute the numbers  $F_v(a_1, \dots, a_s; m - 1)$  when  $\max\{a_1, \dots, a_s\} = 6$ .

*Keywords:* Folkman number, clique number, independence number, chromatic number

**1 Introduction**

In this paper only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:

$V(G)$  - the vertex set of  $G$ ;

$E(G)$  - the edge set of  $G$ ;

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$\overline{G}$  - the complement of  $G$ ;  
 $\omega(G)$  - the clique number of  $G$ ;  
 $\alpha(G)$  - the independence number of  $G$ ;  
 $\chi(G)$  - the chromatic number of  $G$ ;  
 $N_G(v), v \in V(G)$  - the set of all vertices of  $G$  adjacent to  $v$ ;  
 $d(v), v \in V(G)$  - the degree of the vertex  $v$ , i.e.  $d(v) = |N(v)|$ ;  
 $\Delta(G)$  - the maximum degree of  $G$ ;  
 $\delta(G)$  - the minimum degree of  $G$ ;  
 $G - v, v \in V(G)$  - subgraph of  $G$  obtained from  $G$  by deleting the vertex  $v$  and all edges incident to  $v$ ;  
 $G - e, e \in E(G)$  - subgraph of  $G$  obtained from  $G$  by deleting the edge  $e$ ;  
 $G + e, e \in E(\overline{G})$  - supergraph of  $G$  obtained by adding the edge  $e$  to  $E(G)$ .  
 $K_n$  - complete graph on  $n$  vertices;  
 $C_n$  - simple cycle on  $n$  vertices;  
 $r'_0 = r'_0(p)$  - see Theorem 5.1;  
 $r''_0 = r''_0(p)$  - see Theorem 6.1;  
 $G_1 + G_2$  - a graph  $G$  for which:  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2) \cup E'$ , where  $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$ , i.e.  $G$  is obtained by connecting every vertex of  $G_1$  to every vertex of  $G_2$ .  
All undefined terms can be found in [27].

Let  $a_1, \dots, a_s$  be positive integers. The expression  $G \xrightarrow{v} (a_1, \dots, a_s)$  means that for any coloring of  $V(G)$  in  $s$  colors ( $s$ -coloring) there exists  $i \in \{1, \dots, s\}$  such that there is a monochromatic  $a_i$ -clique of color  $i$ . In particular,  $G \xrightarrow{v} (a_1)$  means that  $\omega(G) \geq a_1$ . Further, for convenience instead of  $G \xrightarrow{v} \underbrace{(2, \dots, 2)}_r$  we write  $G \xrightarrow{v} (2_r)$  and instead of  $G \xrightarrow{v} \underbrace{(2, \dots, 2, a_1, \dots, a_s)}_r$  we write  $G \xrightarrow{v} (2_r, a_1, \dots, a_s)$ .

Define:

$$\mathcal{H}(a_1, \dots, a_s; q) = \left\{ G : G \xrightarrow{v} (a_1, \dots, a_s) \text{ and } \omega(G) < q \right\}.$$

$$\mathcal{H}(a_1, \dots, a_s; q; n) = \{ G : G \in \mathcal{H}(a_1, \dots, a_s; q) \text{ and } |V(G)| = n \}.$$

**Remark 1.1.** In the special case  $s = 1$  we have

$$\mathcal{H}(a_1; q; n) = \{ G : a_1 \leq \omega(G) < q \text{ and } |V(G)| = n \}.$$

The vertex Folkman number  $F_v(a_1, \dots, a_s; q)$  is defined by the equality:

$$F_v(a_1, \dots, a_s; q) = \min \{ |V(G)| : G \in \mathcal{H}(a_1, \dots, a_s; q) \}.$$

The graph  $G$  is called an extremal graph in  $\mathcal{H}(a_1, \dots, a_s; q)$  if  $G \in \mathcal{H}(a_1, \dots, a_s; q)$  and  $|V(G)| = F_v(a_1, \dots, a_s; q)$ . We denote by  $\mathcal{H}_{extr}(a_1, \dots, a_s; q)$  the set of all extremal graphs in  $\mathcal{H}(a_1, \dots, a_s; q)$ .

We say that  $G$  is a maximal graph in  $\mathcal{H}(a_1, \dots, a_s; q)$  if  $G \in \mathcal{H}(a_1, \dots, a_s; q)$  but  $G + e \notin \mathcal{H}(a_1, \dots, a_s; q), \forall e \in E(\overline{G})$ , i.e.  $\omega(G + e) \geq q, \forall e \in E(\overline{G})$ .  $G$  is a minimal graph in  $\mathcal{H}(a_1, \dots, a_s; q)$  if  $G \in \mathcal{H}(a_1, \dots, a_s; q)$  but  $G - e \notin \mathcal{H}(a_1, \dots, a_s; q), \forall e \in E(G)$ , i.e.  $G - e \not\xrightarrow{v} (a_1, \dots, a_s), \forall e \in E(G)$ .

For convenience we will also define the following term:

**Definition 1.2.** The graph  $G$  is called a  $(+K_t)$ -graph if  $G + e$  contains a new  $t$ -clique for all  $e \in E(\overline{G})$ .

Obviously,  $G \in \mathcal{H}(a_1, \dots, a_s; q)$  is a maximal graph in  $\mathcal{H}(a_1, \dots, a_s; q)$  if and only if  $G$  is a  $(+K_q)$ -graph.

Folkman proves in [7] that:

$$(1.1) \quad F_v(a_1, \dots, a_s; q) \text{ exists} \Leftrightarrow q > \max \{a_1, \dots, a_s\}.$$

Other proofs of (1.1) are given [6] and [12].

Obviously  $F_v(a_1, \dots, a_s; q)$  is a symmetric function of  $a_1, \dots, a_s$  and if  $a_i = 1$ , then

$$F_v(a_1, \dots, a_s; q) = F_v(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_s; q).$$

Therefore, it is enough to consider only such Folkman numbers  $F_v(a_1, \dots, a_s; q)$  for which

$$(1.2) \quad 2 \leq a_1 \leq \dots \leq a_s$$

We call the numbers  $F_v(a_1, \dots, a_s; q)$ , for which the inequalities (1.2) hold, canonical vertex Folkman numbers.

In [13] for arbitrary positive integers  $a_1, \dots, a_s$  the following are defined

$$(1.3) \quad m(a_1, \dots, a_s) = m = \sum_{i=1}^s (a_i - 1) + 1 \quad \text{and} \quad p = \max \{a_1, \dots, a_s\}.$$

Obviously  $K_m \xrightarrow{v} (a_1, \dots, a_s)$  and  $K_{m-1} \not\xrightarrow{v} (a_1, \dots, a_s)$ . Therefore

$$F_v(a_1, \dots, a_s; q) = m, \quad q \geq m + 1.$$

The following theorem for the numbers  $F_v(a_1, \dots, a_s; m)$  is true:

**Theorem 1.3.** *Let  $a_1, \dots, a_s$  be positive integers and let  $m$  and  $p$  be defined by the equalities (1.3). If  $m \geq p + 1$ , then:*

$$(a) \quad F_v(a_1, \dots, a_s; m) = m + p, \quad [13], [12].$$

$$(b) \quad K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$$

*is the only  $(m + p)$ -vertex(extremal) graph in  $\mathcal{H}(a_1, \dots, a_s; m)$ , [12].*

The condition  $m \geq p + 1$  is necessary according to 1.1. Other proofs of Theorem 1.3 are given in [20] and [21].

Very little is known about the numbers  $F_v(a_1, \dots, a_s; m - 1)$ . According to (1.1) we have

$$(1.4) \quad F_v(a_1, \dots, a_s; m - 1) \text{ exists} \Leftrightarrow m \geq p + 2.$$

The following bounds are known:

$$(1.5) \quad m + p + 2 \leq F_v(a_1, \dots, a_s; m - 1) \leq m + 3p,$$

where the lower bound is true if  $p \geq 2$  and the upper bound is true if  $p \geq 3$ . The lower bound is obtained in [20] and the upper bound is obtained in [10]. In

the border case  $m = p + 2$  the upper bounds in (1.5) are significantly improved in [26].

When  $p = \max \{a_1, \dots, a_s\} \leq 5$  we have

$$(1.6) \quad F_v(a_1, \dots, a_s, m-1) = \begin{cases} m+4, & \text{if } p=2 \text{ and } m \geq 6, [17] \\ m+6, & \text{if } p=3 \text{ and } m \geq 6, [22] \\ m+7, & \text{if } p=4 \text{ and } m \geq 6, [22] \\ m+9, & \text{if } p=5 \text{ and } m \geq 7, [1] \end{cases}$$

In the cases  $p = 2$  and  $p = 3$  we also know the numbers:  $F_v(2, 2, 2; 3) = 11$ , [15] and [3],  $F_v(2, 2, 2, 2; 4) = 11$ , [18] (see also [19]),  $F_v(2, 2, 3; 4) = 14$ , [20] and [4],  $F_v(3, 3; 4) = 14$ , [16] and [23]. These numbers and the numbers (1.6) are all the numbers in the form  $F_v(a_1, \dots, a_s; m-1)$  when  $\max \{a_1, \dots, a_s\} \leq 5$ . We do not know any of these numbers when  $\max \{a_1, \dots, a_s\} \geq 6$ . In [1] we prove that

$$(1.7) \quad m+9 \leq F_v(a_1, \dots, a_s; m-1) \leq m+10,$$

when  $\max \{a_1, \dots, a_s\} = 6$ .

In this paper we complete the computation of these numbers by proving the following

**Main Theorem 1.** *Let  $a_1, \dots, a_s$  be positive integers, such that*

$$2 \leq a_1 \leq \dots \leq a_s = 6,$$

*let  $m = \sum_{i=1}^s (a_i - 1) + 1$  and  $m \geq 8$ . Then*

$$(a) \quad F_v(a_1, \dots, a_s; m-1) = m+9, \text{ if } a_1 = \dots = a_{s-1} = 2.$$

$$(b) \quad F_v(a_1, \dots, a_s; m-1) = m+10, \text{ if } a_{s-1} \geq 3.$$

**Remark 1.4.** *According to (1.4) the condition  $m \geq 8$  is necessary.*

This paper is organized in sections. In the first section the necessary definitions are given, an overview of the known results is provided and at the end we formulate the Main Theorem. In the second section we formulate some auxiliary propositions. In the third section we present some computer algorithms for finding the maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$ . In the fourth section we compute the numbers  $F_v(2, 2, 6; 7)$  and  $F_v(3, 6; 7)$ . First, with the help of a computer we prove that  $|\mathcal{H}(2, 2, 6; 7; 17)| = 3$  (Theorem 4.3) and with the help of this result and (4.2) we obtain  $F_v(2, 2, 6; 7) = 17$  and  $F_v(3, 6; 7) = 18$  (Theorem 4.1). In section 5 we prove the Main Theorem (a), and in section 6 we prove the Main Theorem (b).

## 2 Some auxiliary results

Let  $a_1, \dots, a_s$  be positive integers and  $m = \sum_{i=1}^s (a_i - 1) + 1$ . Obviously, if  $a_i \geq 2$

$$(2.1) \quad G \xrightarrow{v} (a_1, \dots, a_s) \Rightarrow G \xrightarrow{v} (a_1, \dots, a_{i-1}, 2, a_i - 1, a_{i+1}, \dots, a_s).$$

By repeatedly applying (2.1) we obtain

$$(2.2) \quad G \xrightarrow{v} (a_1, \dots, a_s) \Rightarrow G \xrightarrow{v} (2_{m-1}).$$

Since  $G \xrightarrow{v} (2_{m-1}) \Leftrightarrow \chi(G) \geq m$ , from (2.2) it follows

$$(2.3) \quad G \xrightarrow{v} (a_1, \dots, a_s) \Rightarrow \chi(G) \geq m, [21].$$

This fact will be necessary in the proof of Theorem 6.1.

We suppose the following conjecture is true:

**Conjecture 2.1.** *If  $G \in \mathcal{H}_{extr}(a_1, \dots, a_s; m-1)$ , then  $\chi(G) \leq m+1$ .*

For all known examples of extremal graphs this inequality holds.

Let the numbers  $a_1, \dots, a_s$  satisfy the inequalities

$$2 \leq a_1 \leq \dots \leq a_s = p.$$

As before, by repeatedly applying (2.1) we obtain that if  $a_{s-1} \geq 3$

$$(2.4) \quad G \xrightarrow{v} (a_1, \dots, a_s) \Rightarrow G \xrightarrow{v} (2_{m-p-2}, 3, p)$$

and therefore it is true that

$$(2.5) \quad F_v(2_{m-p-2}, 3, p; m-1) \leq F_v(a_1, \dots, a_s; m-1), \text{ if } a_{s-1} \geq 3.$$

Further, we will use the following obvious

**Proposition 2.2.** *Let  $G$  be a graph,  $G \xrightarrow{v} (a_1, \dots, a_s)$  and  $A \subseteq V(G)$  be an independent set. Then if  $a_i \geq 2$*

$$G - A \xrightarrow{v} (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_s).$$

We will also need the following improvement of the lower bound in (1.5)

**Theorem 2.3.** [22] *Let  $a_1, \dots, a_s$  be positive integers, let  $m$  and  $p$  be defined by the equalities (1.3),  $p \geq 3$  and  $m \geq p+2$ . If  $F_v(2, 2, p; p+1) \geq 2p+5$ , then*

$$F_v(a_1, \dots, a_s; m-1) \geq m+p+3.$$

### 3 Algorithms

In this section we present algorithms for finding all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with the help of a computer. The remaining graphs in this set can be obtained by removing edges from the maximal graphs. These algorithms are very similar to the algorithms from [1] and [2]. However, we will present them in detail since they are essential to this paper. The idea for these algorithms comes from [23] (see Algorithm A1). Similar algorithms are used in [4], [28], [11] and [24]. Also with the help of a computer, results for Folkman numbers are obtained in [8], [26], [25] and [5].

The following proposition for maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  will be useful

**Proposition 3.1.** *Let  $G$  be a maximal graph in  $\mathcal{H}(a_1, \dots, a_s; q; n)$ ,  $a_1 \geq 2$ . Let  $v_1, v_2, \dots, v_k$  be independent vertices of  $G$  and  $H = G - \{v_1, v_2, \dots, v_k\}$ . Then:*

(a)  $H \in \mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - k)$

*In the case  $a_1 = 2$ , (a) is equivalent to  $H \in \mathcal{H}(a_2, \dots, a_s; q; n - k)$ .*

(b)  $H$  is a  $(+K_{q-1})$ -graph

(c)  $N_G(v_i)$  is a maximal  $K_{q-1}$ -free subset of  $V(H)$ ,  $i = 1, \dots, k$

*Proof.* The proposition (a) is true according to Proposition 2.2, (b) and (c) follow obviously from the maximality of  $G$ .  $\square$

We will define an algorithm which is based on Proposition 3.1 and generates all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number at least  $k$ .

**Algorithm 3.2.** *Finding all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number at least  $k$  by adding  $k$  independent vertices to the  $(+K_{q-1})$ -graphs in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - k)$ .*

1. Denote by  $\mathcal{A}$  the set of all  $(+K_{q-1})$ -graphs in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - k)$ . The obtained maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  will be output in  $\mathcal{B}$ , let  $\mathcal{B} = \emptyset$ .

2. For each graph  $H \in \mathcal{A}$ :

2.1. Find the family  $\mathcal{M}(H) = \{M_1, \dots, M_t\}$  of all maximal  $K_{q-1}$ -free subsets of  $V(H)$ .

2.2. Consider all the  $k$ -tuples  $(M_{i_1}, M_{i_2}, \dots, M_{i_k})$  of elements of  $\mathcal{M}(H)$ , for which  $1 \leq i_1 \leq \dots \leq i_k \leq t$  (in these  $k$ -tuples some subsets  $M_i$  can coincide). For each such  $k$ -tuple construct the graph  $G = G(M_{i_1}, M_{i_2}, \dots, M_{i_k})$  by adding to  $V(H)$  new independent vertices  $v_1, v_2, \dots, v_k$ , so that  $N_G(v_j) = M_{i_j}$ ,  $j = 1, \dots, k$ . If  $\omega(G + e) = q, \forall e \in E(\overline{G})$ , then add  $G$  to  $\mathcal{B}$ .

3. Exclude the isomorph copies of graphs from  $\mathcal{B}$ .

4. Exclude from  $\mathcal{B}$  all graphs which are not in  $\mathcal{H}(a_1, \dots, a_s; q; n)$ .

Note that in the special case  $s = 1$  it is true that if  $G$  is a maximal graph in  $\mathcal{H}(a_1; q; n)$ , then  $G \in \mathcal{H}(q-1; q; n)$ , and if  $G$  is a  $(+K_{q-1})$  graph in  $\mathcal{H}(a_1-1; q; n)$ , then  $G \in \mathcal{H}(q-2; q; n)$ .

**Theorem 3.3.** *Upon completion of Algorithm 3.2 the obtained set  $\mathcal{B}$  coincides with the set of all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number at least  $k$ .*

*Proof.* From step 4 we see that  $\mathcal{B} \subseteq \mathcal{H}(a_1, \dots, a_s; q; n)$ , and from step 2.2 it becomes clear that  $\mathcal{B}$  contains only maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number at least  $k$ . Let  $G$  be an arbitrary maximal graph in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number at least  $k$ . We will prove that  $G \in \mathcal{B}$ . Let  $v_1, \dots, v_k$  be independent vertices of  $G$  and  $H = G - \{v_1, \dots, v_k\}$ . According to Proposition 3.1(a) and (b),  $H \in \mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - k)$  and  $H$  is a  $(+K_{q-1})$ -graph. Therefore, in step 1 we have  $H \in \mathcal{A}$ . According to Proposition 3.1(c),  $N_G(v_i) \in \mathcal{M}(H)$  for all  $i \in \{1, \dots, k\}$ , hence in step 2.2  $G$  is added to  $\mathcal{B}$ .  $\square$

Let us note that if  $G \in \mathcal{H}(a_1, \dots, a_s; q; n)$  and  $n \geq q$ , then  $G \neq K_n$  and therefore  $\alpha(G) \geq 2$ . In this case, with the help of Algorithm 3.2 we can obtain all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  by adding 2 non-adjacent vertices to the  $(+K_{q-1})$ -graphs in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - 2)$ .

It is clear that if  $G$  is a graph for which  $\alpha(G) = 2$  and  $H$  is a subgraph of  $G$  obtained by removing independent vertices, then  $\alpha(H) \leq 2$ . We modify Algorithm 3.2 in the following way to obtain the maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number 2:

**Algorithm 3.4.** *A modification of Algorithm 3.2 for finding all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number 2 by adding 2 non-adjacent vertices to the  $(+K_{q-1})$ -graphs in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - 2)$  with independence number not greater than 2.*

*In step 1 of Algorithm 3.2 we add the condition that the set  $\mathcal{A}$  contains only the  $(+K_{q-1})$ -graphs  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - 2)$  with independence number not greater than 2, and at the end of step 2.2 after the condition  $\omega(G + e) = q, \forall e \in E(G)$  we also add the condition  $\alpha(G) = 2$ .*

Thus, finding all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number 2 is reduced to finding all  $(+K_{q-1})$ -graphs with independence number not greater than 2 in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - 2)$  and finding the remaining maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  with independence number greater than or equal to 3 is reduced to finding all  $(+K_{q-1})$ -graphs in  $\mathcal{H}(a_1 - 1, a_2, \dots, a_s; q; n - 3)$ . In this way, we can obtain all maximal graphs in  $\mathcal{H}(a_1, \dots, a_s; q; n)$  in steps, starting from graphs with a small number of vertices.

The *nauty* programs [14] have an important role in this work. We use them for fast generation of non-isomorphic graphs and graph isomorph rejection.

## 4 Computation of the numbers $F_v(2, 2, 6; 7)$ and $F_v(3, 6; 7)$

Let  $a_1, \dots, a_s$  be positive integers and let  $m$  and  $p$  be defined by the equalities (1.3). According to (1.4),  $F_v(a_1, \dots, a_s; m - 1)$  exists if and only if  $m \geq p + 2$ . In the border case  $m = p + 2$ ,  $p \geq 3$ , there are only two canonical numbers in the form  $F_v(a_1, \dots, a_s; m - 1)$ , namely  $F_v(2, 2, p; p + 1)$  and  $F_v(3, p; p + 1)$ . The computation of the numbers  $F_v(a_1, \dots, a_s; m - 1)$  when  $\max\{a_1, \dots, a_s\} = 6$ , i.e. the proof of the Main Theorem, will be done with the help of the numbers  $F_v(2, 2, 6; 7)$  and  $F_v(3, 6; 7)$ . Because of this, we will first compute these two numbers by proving

**Theorem 4.1.**  $F_v(2, 2, 6; 7) = 17$  and  $F_v(3, 6; 7) = 18$ .

From (2.1) it is easy to see that

$$G \xrightarrow{v} (3, p) \Rightarrow G \xrightarrow{v} (2, 2, p)$$

and therefore

$$(4.1) \quad F_v(2, 2, p; p + 1) \leq F_v(3, p; p + 1).$$

In [9] the following problem is formulated:

**Problem 4.2.** [9] Does there exist a positive integer  $p$  for which  $F_v(2, 2, p; p + 1) \neq F_v(3, p; p + 1)$ ?

Theorem 4.1 gives a positive answer to Problem 4.2. Since

$$F_v(2, 2, p; p + 1) = F_v(3, p; p + 1), \quad p \leq 5$$

(see [1]), it becomes clear that  $p = 6$  is the smallest positive integer for which

$$F_v(2, 2, p; p + 1) \neq F_v(3, p; p + 1)$$

For the proof of Theorem 4.1 we will need

**Theorem 4.3.**  $|\mathcal{H}(2, 2, 6; 7; 17)| = 3$  and  $\mathcal{H}_{extr}(2, 2, 6; 7) = \mathcal{H}(2, 2, 6; 7; 17) = \{G_1, G_2, G_3\}$  (see Figure 1). Some properties of the graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  are given in Table 1.

*Proof.* We will find all graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  with the help of a computer.

First, we will prove that there are not any maximal graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  with independence number greater than 2. It is clear that  $K_6$  and  $K_6 - e$  are the only  $(+K_6)$ -graphs in  $\mathcal{H}(3; 7; 6)$ . With the help of Algorithm 3.2 we add 2 non-adjacent vertices to these graphs to find all maximal graphs in  $\mathcal{H}(4; 7; 8)$ . By removing edges from them we find all  $(+K_6)$ -graphs in  $\mathcal{H}(4; 7; 8)$ . In the same way, we successively obtain all maximal and all  $(+K_6)$ -graphs in the sets:  $\mathcal{H}(5; 7; 10)$ ,  $\mathcal{H}(6; 7; 12)$ ,  $\mathcal{H}(2, 6; 7; 14)$ .

According to Theorem 3.3, we obtain all maximal graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  with independence number greater than 2 by adding 3 independent vertices to the  $(+K_6)$ -graphs in  $\mathcal{H}(2, 6; 7; 14)$  with the help of Algorithm 3.2. Since the obtained set of graphs  $\mathcal{B}$  is empty, we conclude that the independence number of the maximal graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  is not greater than 2.

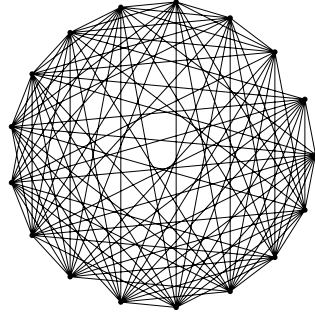
After that, we find the maximal graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  with independence number 2. It is clear that  $K_7 - e$  is the only maximal graph in  $\mathcal{H}(3; 7; 7)$ . By removing edges from it we obtain all  $(+K_6)$ -graphs in  $\mathcal{H}(3; 7; 7)$  with independence number 2. With the help of Algorithm 3.4 we add 2 non-adjacent vertices to these graphs to find all maximal graphs in  $\mathcal{H}(4; 7; 9)$  with independence number 2. By removing edges from them we find all  $(+K_6)$ -graphs in  $\mathcal{H}(4; 7; 9)$  with independence number 2. In the same way, we successively obtain all maximal and all  $(+K_6)$ -graphs with independence number 2 in the sets:  $\mathcal{H}(5; 7; 11)$ ,  $\mathcal{H}(6; 7; 13)$ ,  $\mathcal{H}(2, 6; 7; 15)$ ,  $\mathcal{H}(2, 2, 6; 7; 17)$ .

The number of graphs obtained in each step is shown in Table 2. There is only one maximal graph in  $\mathcal{H}(2, 2, 6; 7; 17)$ . It is the graph  $G_1$  shown on Figure 1. By removing edges from this graph and checking for membership in  $\mathcal{H}(2, 2, 6; 7; 17)$  we find that there are only two other graphs in this set, which we will denote by  $G_2$  and  $G_3$  (see Figure 1). Let us also note, that  $G_1 \supset G_2 \supset G_3$  and for the graphs  $G_1$ ,  $G_2$  and  $G_3$  the inequality (2.3) is strict (see Conjecture 2.1). It is clear that  $G_3$  is the only minimal graph in  $\mathcal{H}(2, 2, 6; 7; 17)$ . From (1.7) ( $m = 8, p = 6$ ) and (4.1) we obtain

$$(4.2) \quad 17 \leq F_v(2, 2, 6; 7) \leq F_v(3, 6; 7) \leq 18.$$

Let us note, that the inequality  $F_v(3, 6; 7) \leq 18$  is first proved in [26]. From (4.2) it follows  $\mathcal{H}_{extr}(2, 2, 6; 7) = \mathcal{H}(2, 2, 6; 7; 17) = \{G_1, G_2, G_3\}$ . Thus, we finish the proof of Theorem 4.3. □



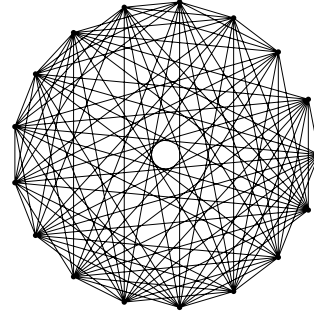


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$G_1$

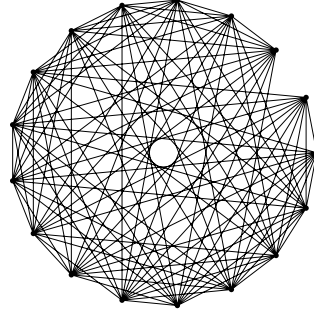


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$G_2$



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1 1 0 1 1 1 1 1 0 1 1 1 0 1 1 1 1 1 1 0

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$G_3$

Figure 1: All graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$

Graph	$ E(G) $	$\delta(G)$	$\Delta(G)$	$\alpha(G)$	$\chi(G)$	$ Aut(G) $
$G_1$	108	12	13	2	9	2
$G_2$	107	11	13	2	9	4
$G_3$	106	11	13	2	9	40

Table 1: The graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$  and some of their properties

### Proof of Theorem 4.1

The equality  $F_v(2, 2, 6; 7) = 17$  follows from Theorem 4.3. With the help of a computer we check that the graphs from Theorem 4.3 do not belong in  $\mathcal{H}(3, 6; 7)$ . Since  $G_1 \supset G_2 \supset G_3$ , it is enough to check that  $G_1 \notin \mathcal{H}(3, 6; 7)$ . Since  $\mathcal{H}(3, 6; 7) \subseteq \mathcal{H}(2, 2, 6; 7)$  (see (2.1)) we come to the conclusion that  $H_v(3, 6; 7; 17) = \emptyset$  and from (4.2) we obtain  $F_v(3, 6; 7) = 18$ .

## 5 Proof of Main Theorem (a)

We will do the proof with the help of the following

**Theorem 5.1.** [1] *Let  $r'_0(p) = r'_0$  be the smallest non-negative integer for which*

$$\min_{r \geq 2} \{F_v(2_r, p; r + p - 1) - r\} = F_v(2_{r'_0}, p; r'_0 + p - 1) - r'_0.$$

*Then:*

(a)  $F_v(2_r, p; r + p - 1) = F_v(2_{r'_0}, p; r'_0 + p - 1) + r - r'_0, \quad r \geq r'_0.$

(b) *if  $r'_0 = 2$ , then*

$$F_v(2_r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2$$

(c) *if  $r'_0 > 2$  and  $G$  is an extremal graph in  $\mathcal{H}(2_{r'_0}, p; r'_0 + p - 1)$ , then*

$$G \xrightarrow{v} (2, r'_0 + p - 2).$$

(d)  $r'_0 < F_v(2, 2, p; p + 1) - 2p.$

Theorem 5.1 is proved in [1] as Theorem 5.2. We will note that the proof of Theorem 5.1 is analogous to that of Theorem 6.1 from this paper.

In relation to Theorem 5.1(b) in [1] we formulate

**Conjecture 5.2.** [1] *If  $p \geq 4$ , then  $r'_0(p) = 2$  and therefore according to Theorem 5.1(b)*

$$F_v(2_r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2.$$

It is not difficult to see that Conjecture 5.2 is true if and only if the sequence  $\{F_v(2_r, p; r + p - 1)\}$  for fixed  $p$  is strictly increasing with respect to  $r \geq 2$ . Conjecture 5.2 is true when  $p = 4$  and  $p = 5$  (see Remark 5.6). We will prove that when  $p = 6$  Conjecture 5.2 is also true. More specifically, we will prove

**Theorem 5.3.**  $r'_0(6) = 2$  and therefore  $F_v(2_r, 6; r + 5) = r + 15$ ,  $r \geq 2$ .

Before proving Theorem 5.3 we will prove

**Theorem 5.4.** Let  $F_v(2, 2, p; p + 1) \leq 2p + 5$ . Then  $r'_0(p) = 2$  and

$$F_v(2_r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2.$$

*Proof.* From Theorem 5.1(b) it follows that it is enough to prove the equality  $r'_0(p) = 2$ . According to (1.5),  $F_v(2, 2, p; p + 1) \geq 2p + 4$ . Therefore, only the following two cases are possible:

*Case 1.*  $F_v(2, 2, p; p + 1) = 2p + 4$ . According to (1.5)

$$F_v(2_r, p; r + p - 1) \geq m + p + 2 = r + 2p + 2.$$

Therefore,

$$F_v(2_r, p; r + p - 1) - r \geq 2p + 2 = F_v(2, 2, p; p + 1) - 2, \quad r \geq 2,$$

and we have  $r'_0(p) = 2$ .

*Case 2.*  $F_v(2, 2, p; p + 1) = 2p + 5$ . From Theorem 2.3 we have  $F_v(2_r, p; r + p - 1) \geq r + 2p + 3$ ,  $r \geq 2$ . From this inequality we obtain

$$F_v(2_r, p; r + p - 1) - r \geq 2p + 3 = F_v(2, 2, p; p + 1) - 2, \quad r \geq 2.$$

Therefore, in this case we also have  $r'_0(p) = 2$ .  $\square$

**Remark 5.5.** It is unknown whether the first case is possible, i.e. if  $F_v(2, 2, p; p + 1) = 2p + 4$  for some  $p$ . If  $p \leq 6$  this equality is not true.

**Remark 5.6.** Since  $F_v(2, 2, 3; 4) = 14$  [20] and [4],  $F_v(a_1, \dots, a_s; m - 1) = m + 6$ , if  $p = 3$  and  $m \geq 6$  [22], it follows that  $r'_0(3) = 3$ . Since  $F_v(2, 2, 4; 5) = 13$  [22], from Theorem 5.4 it follows that  $r'_0(4) = 2$ . The equality  $r'_0(5) = 2$  is also true, but it does not follow from Theorem 5.4. It is proved with the help of a computer in [1] as Theorem 6.1.

### Proof of Theorem 5.3

According to Theorem 4.1,  $F_v(2, 2, 6; 7) = 17$ . From this fact and Theorem 5.4 we obtain  $r'_0(6) = 2$  and the equality  $F_v(2_r, 6; r + 5) = r + 15$ ,  $r \geq 2$ .

### Proof of Main Theorem (a)

Since  $a_1 = \dots = a_{s-1} = 2$  and  $a_s = 6$  we have  $m = s + 5$  and therefore

$$F_v(a_1, \dots, a_s; m - 1) = F_v(2_{s-1}, 6; m - 1) = F_v(2_{m-6}, 6; m - 1).$$

From Theorem 5.3 it now follows that  $F_v(a_1, \dots, a_s; m - 1) = m + 9$ .

## 6 Proof of Main Theorem (b)

We will need the following

**Theorem 6.1.** Let  $r''_0(p) = r''_0$  be the smallest positive integer for which

$$\min \{F_v(2_r, 3, p; r + p + 1) - r\} = F_v(2_{r''_0}, 3, p; r''_0 + p + 1) - r''_0$$

Then

$$(a) \quad F_v(2_r, 3, p; r + p + 1) = F_v(2_{r''_0}, 3, p; r''_0 + p + 1) + r - r''_0, \quad r \geq r''_0.$$

(b) if  $r''_0 = 0$ , then

$$F_v(2_r, 3, p; r + p + 1) = F_v(3, p; p + 1) + r, \quad r \geq 0$$

(c) if  $r''_0 > 0$  and  $G$  is an extremal graph in  $\mathcal{H}(2_{r''_0}, 3, p; r''_0 + p + 1)$ , then

$$G \xrightarrow{v} (2, r''_0 + p).$$

(d)  $r''_0 < F_v(3, p; p + 1) - 2p - 2$ .

*Proof.* (a) According to the definition of  $r''_0 = r''_0(p)$  we have

$$F_v(2_r, 3, p; r + p + 1) \geq F_v(2_{r''_0}, 3, p; r''_0 + p + 1) + r - r''_0, \quad r \geq 0.$$

Now we will prove that if  $r \geq r''_0$  the opposite inequality is also true. Let us note that if  $G \xrightarrow{v} (a_1, \dots, a_s)$ , then  $K_1 + G \xrightarrow{v} (2, a_1, \dots, a_s)$ . It follows

$$(6.1) \quad G \xrightarrow{v} (a_1, \dots, a_s) \Rightarrow K_t + G \xrightarrow{v} (2_t, a_1, \dots, a_s).$$

Let  $G \in \mathcal{H}_{extr}(2_{r''_0}, 3, p; r''_0 + p + 1)$ . Then from (6.1) it follows that  $K_{r-r''_0} + G \in \mathcal{H}(2_r, 3, p; r + p + 1)$ ,  $r \geq r''_0$ . Therefore

$$F_v(2_r, 3, p; r + p + 1) \leq |V(K_{r-r''_0} + G)| = F_v(2_{r''_0}, 3, p; r''_0 + p + 1) + r - r''_0, \quad r \geq r''_0.$$

Thus, (a) is proved.

(b) If  $r''_0(p) = 0$ , then obviously the equality (b) follows from (a).

(c) Assume the opposite is true and let  $G$  be an extremal graph in  $\mathcal{H}(2_{r''_0}, 3, p; r''_0 + p + 1)$ , such that  $V(G) = V_1 \cup V_2$  where  $V_1$  is an independent set and  $V_2$  does not contain  $(r''_0 + p)$ -clique. We can assume that  $V_1 \neq \emptyset$ . Let  $G_1 = G[V_2]$ . Then according to (2.2),  $G_1 \xrightarrow{v} (2_{r''_0-1}, 3, p)$ . Therefore  $G_1 \in \mathcal{H}(2_{r''_0-1}, 3, p; r''_0 + p)$  and

$$|V(G)| - 1 \geq |V(G_1)| \geq F_v(2_{r''_0-1}, 3, p; r''_0 + p).$$

Since  $|V(G)| = F_v(2_{r''_0}, 3, p; r''_0 + p + 1)$  we obtain

$$F_v(2_{r''_0-1}, 3, p; r''_0 + p) - (r''_0 - 1) \leq F_v(2_{r''_0}, 3, p; r''_0 + p + 1) - r''_0,$$

which contradicts the definition of  $r''_0$ .

(d) According to (1.5)  $F_v(3, p; p + 1) \geq 2p + 4$  and therefore in the case  $r''_0 = 0$  the inequality holds. Let  $r''_0 > 0$  and  $G$  be an extremal graph in  $\mathcal{H}(2_{r''_0}, 3, p; r''_0 + p + 1)$ . According to (2.3)

$$(6.2) \quad \chi(G) \geq r''_0 + p + 2.$$

From (c) and Theorem 1.3 it follows that

$$|V(G)| \geq 2r''_0 + 2p + 1.$$

Since  $\chi(\overline{C}_{2r''_0+2p+1}) = r''_0 + p + 1$ , from (6.2) it follows  $G \neq \overline{C}_{2r''_0+2p+1}$ . From Theorem 1.3(b) now follows

$$|V(G)| = F_v(2_{r''_0}, 3, p; r''_0 + p + 1) \geq 2r''_0 + 2p + 2.$$

Since  $r''_0 > 0$ , we have

$$F_v(2_{r''_0}, 3, p; r''_0 + p + 1) - r''_0 < F_v(3, p; p + 1).$$

From these inequalities we can easily see that

$$r''_0 < F_v(3, p; p + 1) - 2p - 2. \quad \square$$

We suppose the following conjecture is true

**Conjecture 6.2.**  $r_0''(p) = 0$ ,  $p \geq 4$ , and therefore according to Theorem 6.1(b)

$$F_v(2_r, 3, p; r + p + 1) = F_v(3, p; p + 1) + r.$$

Since  $F_v(2, 2, 3; 4) = 14$  and  $F_v(3, 3; 4) = 14$ , from (1.6) we obtain  $r_0''(3) = 1$ . It is not difficult to see that Conjecture 6.2 is true if and only if the sequence  $\{F_v(2_r, 3, p; r + p + 1)\}$  for fixed  $p$  is strictly increasing with respect to  $r$ . Therefore, from (1.6) we see that  $r_0''(4) = 0$  and  $r_0''(5) = 0$ . We will prove that when  $p = 6$  Conjecture 6.2 is also true. The Main Theorem (b) follows easily from this fact.

**Theorem 6.3.**  $r_0''(6) = 0$ .

*Proof.* From Theorem 6.1(d) and  $F_v(3, 6; 7) = 18$  we obtain  $r_0''(6) < 4$ . Therefore we have to prove  $r_0''(6) \neq 1$ ,  $r_0''(6) \neq 2$  and  $r_0''(6) \neq 3$ . Since  $F_v(3, 6; 7) = 18$ , we have to prove the inequalities  $F_v(2, 3, 6; 8) > 18$ ,  $F_v(2, 2, 3, 6; 9) > 19$  and  $F_v(2, 2, 2, 3, 6; 10) > 20$ , i.e. we have to prove  $\mathcal{H}(2_r, 3, 6; r + 7; r + 17) = \emptyset$ ,  $r = 1, 2, 3$ . From (6.1) ( $t = 1$ ) it is easy to see that  $F_v(2_{r-1}, 3; p) + 1 \geq F_v(2_r, 3; p)$  and therefore it is enough to prove  $F_v(2, 2, 2, 3, 6; 10) > 20$ . However, we also prove the other two inequalities with the help of a computer, because in this way we obtain more information, which is presented in Appendix A.

First, we find the maximal graphs in  $\mathcal{H}(2_r, 3, 6; r + 7; r + 17)$  with independence number greater than 2. It is clear that  $K_{r+6}$  and  $K_{r+6} - e$  are the only  $(+K_{r+6})$ -graphs in  $\mathcal{H}(r + 3; r + 7; r + 6)$ . Starting from them with the help of Algorithm 3.2 we successively obtain all maximal and all  $(+K_{r+6})$ -graphs in the sets:

$(r = 1)$	$(r = 2)$	$(r = 3)$
$\mathcal{H}(5; 8; 9)$	$\mathcal{H}(6; 9; 10)$	$\mathcal{H}(2, 6; 10; 11)$
$\mathcal{H}(6; 8; 11)$	$\mathcal{H}(2, 6; 9; 12)$	$\mathcal{H}(3, 6; 10; 13)$
$\mathcal{H}(2, 6; 8; 13)$	$\mathcal{H}(3, 6; 9; 14)$	$\mathcal{H}(2, 3, 6; 10; 15)$
$\mathcal{H}(3, 6; 8; 15)$	$\mathcal{H}(2, 3, 6; 9; 16)$	$\mathcal{H}(2, 2, 3, 6; 10; 17)$

In the end, with the help of Algorithm 3.2 we add 3 independent vertices to the obtained  $(+K_{r+6})$ -graphs in  $\mathcal{H}(2_{r-1}, 3, 6; r + 7; r + 14)$  to find all maximal graphs in  $\mathcal{H}(2_r, 3, 6; r + 7; r + 17)$  with independence number greater than 2.

After that, we find the maximal graphs in  $\mathcal{H}(2_r, 3, 6; r + 7; r + 17)$  with independence number 2. It is clear that  $K_{r+7} - e$  is the only maximal graph in  $\mathcal{H}(r + 3; r + 7; r + 7)$ . By removing edges from it we obtain all  $(+K_{r+6})$ -graphs in  $\mathcal{H}(r + 3; r + 7; r + 7)$  with independence number 2. Starting from them with the help of Algorithm 3.4 we successively obtain all maximal and all  $(+K_{r+6})$ -graphs with independence number 2 in the sets:

$(r = 1)$	$(r = 2)$	$(r = 3)$
$\mathcal{H}(5; 8; 10)$	$\mathcal{H}(6; 9; 11)$	$\mathcal{H}(2, 6; 10; 12)$
$\mathcal{H}(6; 8; 12)$	$\mathcal{H}(2, 6; 9; 13)$	$\mathcal{H}(3, 6; 10; 14)$
$\mathcal{H}(2, 6; 8; 14)$	$\mathcal{H}(3, 6; 9; 15)$	$\mathcal{H}(2, 3, 6; 10; 16)$
$\mathcal{H}(3, 6; 8; 16)$	$\mathcal{H}(2, 3, 6; 9; 17)$	$\mathcal{H}(2, 2, 3, 6; 10; 18)$
$\mathcal{H}(2, 3, 6; 8; 18)$	$\mathcal{H}(2, 2, 3, 6; 9; 19)$	$\mathcal{H}(2, 2, 2, 3, 6; 10; 20)$

The numbers of graphs obtained in each step are shown in Table 3, Table 4 and Table 5 respectively. In both cases we do not obtain any maximal graphs in the sets  $\mathcal{H}(2_r, 3, 6; r + 7; r + 17)$ ,  $r = 1, 2, 3$  and therefore we have  $F_v(2, 3, 6; 8) > 18$ ,  $F_v(2, 2, 3, 6; 9) > 19$ ,  $F_v(2, 2, 2, 3, 6; 10) > 20$  and  $r_0''(6) = 0$ .  $\square$

## Proof of Main Theorem (b)

According to Theorem 6.3 and Theorem 6.1 we have  $F_v(2_{m-8}, 3, 6; m-1) = m+10$ . From (2.5) it now follows  $F_v(a_1, \dots, a_s; m-1) \geq m+10$ . The opposite inequality is true according to (1.7).

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## A Results of computations

set	independence number	maximal graphs	( $+K_6$ )- graphs
$\mathcal{H}(3; 7; 6)$	-		2
$\mathcal{H}(4; 7; 8)$	-	2	13
$\mathcal{H}(5; 7; 10)$	-	8	324
$\mathcal{H}(6; 7; 12)$	-	56	104 271
$\mathcal{H}(2, 6; 7; 14)$	-	20	5 293
$\mathcal{H}(2, 2, 6; 7; 17)$	$\geq 3$	0	
$\mathcal{H}(3; 7; 7)$	$\leq 2$	1	3
$\mathcal{H}(4; 7; 9)$	$= 2$	2	22
$\mathcal{H}(5; 7; 11)$	$= 2$	5	468
$\mathcal{H}(6; 7; 13)$	$= 2$	24	97 028
$\mathcal{H}(2, 6; 7; 15)$	$= 2$	473	10 018 539
$\mathcal{H}(2, 2, 6; 7; 17)$	$= 2$	1	
$\mathcal{H}(2, 2, 6; 7; 17)$	-	1	

Table 2: Steps in finding all maximal graphs in  $\mathcal{H}(2, 2, 6; 7; 17)$

set	independence number	maximal graphs	( $+K_7$ )- graphs
$\mathcal{H}(4; 8; 7)$	-		2
$\mathcal{H}(5; 8; 9)$	-	2	13
$\mathcal{H}(6; 8; 11)$	-	8	326
$\mathcal{H}(2, 6; 8; 13)$	-	56	105 125
$\mathcal{H}(3, 6; 8; 15)$	-	21	3 459
$\mathcal{H}(2, 3, 6; 8; 18)$	$\geq 3$	0	
$\mathcal{H}(4; 8; 8)$	$\leq 2$	1	3
$\mathcal{H}(5; 8; 10)$	$= 2$	2	22
$\mathcal{H}(6; 8; 12)$	$= 2$	5	489
$\mathcal{H}(2, 6; 8; 14)$	$= 2$	25	119 126
$\mathcal{H}(3, 6; 8; 16)$	$= 2$	509	3 582 157
$\mathcal{H}(2, 3, 6; 8; 18)$	$= 2$	0	
$\mathcal{H}(2, 3, 6; 8; 18)$	-	0	

Table 3: Steps in finding all maximal graphs in  $\mathcal{H}(2, 3, 6; 8; 18)$



set	independence number	maximal graphs	$(+K_8)$ - graphs
$\mathcal{H}(5; 9; 8)$	-		2
$\mathcal{H}(6; 9; 10)$	-	2	13
$\mathcal{H}(2, 6; 9; 12)$	-	8	327
$\mathcal{H}(3, 6; 9; 14)$	-	56	105 281
$\mathcal{H}(2, 3, 6; 9; 16)$	-	21	3 460
$\mathcal{H}(2, 2, 3, 6; 9; 19)$	$\geq 3$	0	
$\mathcal{H}(5; 9; 9)$	$\leq 2$	1	3
$\mathcal{H}(6; 9; 11)$	$= 2$	2	22
$\mathcal{H}(2, 6; 9; 13)$	$= 2$	5	496
$\mathcal{H}(3, 6; 9; 15)$	$= 2$	25	121 499
$\mathcal{H}(2, 3, 6; 9; 17)$	$= 2$	512	3 585 530
$\mathcal{H}(2, 2, 3, 6; 9; 19)$	$= 2$	0	
$\mathcal{H}(2, 2, 3, 6; 9; 19)$	-	0	

Table 4: Steps in finding all maximal graphs in  $\mathcal{H}(2, 2, 3, 6; 9; 19)$

set	independence number	maximal graphs	$(+K_9)$ - graphs
$\mathcal{H}(6; 10; 9)$	-		2
$\mathcal{H}(2, 6; 10; 11)$	-	2	13
$\mathcal{H}(3, 6; 10; 13)$	-	8	327
$\mathcal{H}(2, 3, 6; 10; 15)$	-	56	105 314
$\mathcal{H}(2, 2, 3, 6; 10; 17)$	-	21	3 460
$\mathcal{H}(2, 2, 2, 3, 6; 10; 20)$	$\geq 3$	0	
$\mathcal{H}(6; 10; 10)$	$\leq 2$	1	3
$\mathcal{H}(2, 6; 10; 12)$	$= 2$	2	22
$\mathcal{H}(3, 6; 10; 14)$	$= 2$	5	498
$\mathcal{H}(2, 3, 6; 10; 16)$	$= 2$	25	121 864
$\mathcal{H}(2, 2, 3, 6; 10; 18)$	$= 2$	512	3 585 546
$\mathcal{H}(2, 2, 2, 3, 6; 10; 20)$	$= 2$	0	
$\mathcal{H}(2, 2, 2, 3, 6; 10; 20)$	-	0	

Table 5: Steps in finding all maximal graphs in  $\mathcal{H}(2, 2, 2, 3, 6; 10; 20)$